A PROBLEM ON THE STABILITY OF A SPHERICAL GYROSCOPE

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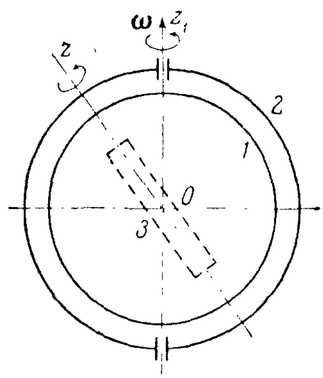
Under certain assumptions on the nature of the viscous gaseous shell, we consider the stability of the vertical rotation of a spherical gyroscope. By the Liapunov-Chetaev method we obtain sufficient conditions for asymptotic stability. The cases of astatic and heavy gyroscopes, and also the case of rotation with variable angular velocity, are treated in particular.

The principal scheme for the motion of a spherical gyroscope was given in [1]. A steel sphere 1 (Figure) is placed inside a closed spherical bowl 2 rotating with constant angular velocity around a stationary vertical axis Oz_1. Gas supplied through special holes in the wall of the bowl forms a shell completely enveloping the sphere and isolating it from the wall of the bowl. It is assumed that at a certain instant the gaseous shell becomes a homogeneous shell with constant thickness (equal to the difference between the radii of the bowl and the sphere) and remains thus for all subsequent time of the motion. Thus, the geometric center of the sphere coincides, by hypotheses, with the center of the bowl and, consequently, is a stationary point. Since the gas has a specific viscosity, the bowl in spinning carries along with it the gaseous shell which in its turn makes the sphere rotate. The sphere has inside it a cylindrical groove 3 for creating a definite dynamic axis of symmetry Oz which at the initial instant coincides with the vertical Oz_1. The ellipsoid of inertia of the sphere with respect to the point O will be the ellipsoid of rotation.

1. The balanced gyroscope. The problem consists of investigating the stability of the established motion of the sphere as a rigid body in which the center of gravity coincides with the point of suspension O and which is acted upon by a rotating moment $\mathbf{M}$ originating in the viscous gaseous shell and proportional to the difference between the angular velocities of the sphere and the bowl. The moment $\mathbf{M}$ is taken in the form



$$\mathbf{M} = -K(\Omega - \omega), \qquad K = \frac{8\pi R^4 \nu}{3d} \qquad (1.1)$$

where Ω is the instantaneous angular velocity of the sphere, ω is the constant angular velocity of the bowl, R is the radius of the sphere, d is the thickness of the gaseous shell and ν is the coefficient of viscosity of the gas.

The results of solving certain analogous problems in hydroaeromechanics [2] prove that our choice of such a numerical value for coefficient K is valid.

Let us note that here the motion of the gas is not considered and that the gaseous shell is mentioned only in connection with the creation of the rotating moment $\mathbf{M}$ with respect to point O. The action of moment $\mathbf{M}$ is analogous to the action of an induction motor, maintaining a constant angular velocity in balanced motion. Let us write the equations of motion of the sphere in the form of the dynamic equations of Euler and the kinematic equations of Poisson

$$\begin{aligned} &A\frac{dp}{dt} + (C - A)\,qr = K(\omega\gamma - p), && \frac{d\gamma}{dt} + q\gamma'' - r\gamma' = 0 \\ &A\frac{dq}{dt} + (A - C)\,pr = K(\omega\gamma' - q), && \frac{d\gamma'}{dt} + r\gamma - p\gamma'' = 0 \qquad (1.2) \\ &C\frac{dr}{dt} = K(\omega\gamma'' - r), && \frac{d\gamma''}{dt} + p\gamma' - q\gamma = 0 \end{aligned}$$

Here A is the equatorial moment of inertia of the sphere, C is the axial moment of inertia of the sphere, p, q and r are the projections of the instantaneous angular velocity Ω of the sphere onto the principal axes of inertia $Oxyz$, γ, γ' and γ'' are the direction cosines of the vertical Oz_1 with respect to the axes Ox, Oy and Oz, respectively. The stationary state under investigation will be the particular solution of system (1.2)

$$p = q = 0, \qquad r = \omega = \text{const}, \qquad \gamma = \gamma' = 0, \qquad \gamma'' = 1 \qquad (1.3)$$

which corresponds to a rotation of the sphere and the bowl as a whole around the vertical axis Oz_1. The right-hand sides of equations (1.2), depending on p, q, r, γ, γ' and γ'', show that the mechanical system whose motion they describe is essentially nonconservative. The system of equations (1.2) allows of only one known integral

$$\gamma^2 + \gamma'^2 + \gamma''^2 = \text{const} \tag{1.4}$$

Let us determine the stability of motion (1.3) in the sense of Liapunov with respect to the variables p, q, r, γ, γ' and γ''.

The problem of stability of vertical rotation (1.3) with respect to p, q, r, γ, γ' and γ'' for a heavy rigid body with a fixed point in the Lagrange case, was solved, as is well known, by Chetaev [3] by the method of constructing Liapunov functions in the form of a linear combination of the integrals of the equations of perturbed motion. In the given case the number of integrals is not sufficient for the construction of a linear combination; however, we can try to construct a sign-definite combination from specially selected functions such that the conditions of Liapunov's stability theorem are satisfied.

Let us introduce into consideration the kinetic energy of the sphere, the projection of the angular momentum $\mathbf{G}$ of the sphere onto axis Oz_1, and the function Φ

$$T = \tfrac{1}{2}(Ap^2 + Aq^2 + Cr^2), \qquad G_{z_1} = Ap\gamma + Aq\gamma' + Cr\gamma''$$
$$\Phi = \gamma^2 + \gamma'^2 + \gamma''^2 \tag{1.5}$$

By denoting the perturbed motion corresponding to (1.3) by

$$p = \xi, \quad q = \eta, \quad r = \omega + \zeta, \quad \gamma = \alpha, \quad \gamma' = \beta, \quad \gamma'' = 1 + \delta \tag{1.6}$$

we can write

$$\begin{aligned} F_1 &= A\,(\xi^2 + \eta^2) + C\zeta^2 + 2C\omega\zeta \\ F_2 &= A\,(\xi\alpha + \eta\beta) + C\zeta\delta + C\,(\omega\delta + \zeta) \\ F_3 &= \alpha^2 + \beta^2 + \delta^2 + 2\delta \end{aligned} \tag{1.7}$$

Let us make a linear combination of functions (1.7) in the form

$$2V = F_1 - 2\omega F_2 + C\omega^2 F_3$$

It is not difficult to see that $2V$ will be a quadratic form

$$2V = A\,(\xi^2 + \eta^2) - 2\omega A\,(\xi\alpha + \eta\beta) + C\omega^2\,(\alpha^2 + \beta^2) + C\,(\zeta - \omega\delta)^2 \tag{1.8}$$

The derivative of function V taken relative to the equations of perturbed motion corresponding to motion (1.3)

$$V^{\cdot} = -K[(\xi - \omega\alpha)^2 + (\eta - \omega\beta)^2 + (\zeta - \omega\delta)^2] \tag{1.9}$$

will be sign-constant of a sign opposite to that of V. The function V, not being sign-definite in all of the variables ξ, η, ζ, α, β and δ, will be sign-definite in the variables ξ, η, α, β and $\delta - \omega\zeta$ when the condition $C > A$ is satisfied, which indicates that in this case the axis Oz will be the minor axis of the ellipsoid of inertia of the sphere with respect to the point O. Thus, on the basis of the theorem on stability with respect to a portion of the variables [4], motion (1.3) is stable with respect to p, q, γ, γ' and $r - \omega\gamma''$. The variables γ, γ' and γ'' are connected during the motion by relation (1.4), and therefore, from the stability of solution (1.3) with respect to γ and γ' there follows stability also with respect to γ'', and from stability with respect to γ'' and $r - \omega\gamma''$ follows stability with respect to r. Consequently, under the condition $C > A$, motion (1.3) will be stable with respect to all the variables p, q, r, γ, γ' and γ''.

The derivative $V^{\cdot}$ is not a sign-definite function, therefore, in general, it is not possible to infer the asymptotic stability of state (1.3). However, in certain cases the sign-constancy of the derivative is sufficient for the unperturbed motion under study to be asymptotically stable. Such a sufficient criterion for asymptotic stability is given by the theorem stated in [5]. On the basis of this theorem the asymptotic stability of motion (1.3) will be demonstrated if we can establish that the region $V^{\cdot} = 0$ does not wholly contain the solutions of the equations of perturbed motion corresponding to (1.3), except the unperturbed motion (1.3) itself. Let us clarify in detail the nature of the region $V^{\cdot} = 0$. If in equality (1.9) we pass from perturbations to values of the original variables of the perturbed motion in accordance with (1.6), we obtain

$$V^{\cdot} = -K[(p-\omega\gamma)^2 + (q-\omega\gamma')^2 + (r-\omega\gamma'')^2] = -K|\Omega - \boldsymbol{\omega}|^2 \tag{1.10}$$

Hence it is seen that the derivative $V^{\cdot}$ is proportional to the square of the modulus of the vector difference between the angular velocities of the sphere in perturbed and unperturbed motions, and consequently, equals zero when $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

System (1.2) shows that the sphere cannot accomplish established motions – rotations around the vertical with constant angular velocity $\boldsymbol{\omega}$ – except motion (1.3). Hence, the question whether the region $V^{\cdot} = 0$ does or does not contain the entire integral curves of the equations of perturbed motion corresponding to (1.3), reduces to the question of whether the sphere for specified perturbations can or cannot accomplish a continuous sequence of instantaneous rotations with some angular velocity $\boldsymbol{\omega}$. In other words, can or cannot the rigid body move such that the fixed axoid degenerates to some fixed straight line at the same time that the moving axoid remains a nondegenerate cone? The kinematics of rigid bodies gives a negative answer to this question [6], which allows us to infer that under the condition $C > A$ motion (1.3) will be asymptotically stable independently of the magnitude of K. Consequently, for achieving asymptotic stability of vertical rotation (1.3) of the sphere, it is sufficient that the axis of rotation be the minor axis of the ellipsoid of inertia of the sphere with respect to the point of suspension O. An analogous result was obtained in [7] by investigating the linear problem.

Note. The results of Section 1 can be extended if we waive the requirement of symmetry of the rigid body. Let A, B and C be the moments of inertia of the body with respect to its principal axes $Oxyz$. The other assumptions of Section 1 remain in force. It is not difficult to show that the function V

$$2V = A\xi^2 + B\eta^2 - 2\omega\ (A\xi\alpha + B\eta\beta) + C\omega^2(\alpha^2 + \beta^2) + C(\zeta - \omega\delta)^2 \tag{1.11}$$

positive definite in the variables ξ, η, α, β and $\zeta - \omega\delta$ when the inequalities

$$C > A, \quad C > B \tag{1.12}$$

are satisfied, has the derivative (1.9) relative to the equations of perturbed motion corresponding to state (1.3). Hence, the unperturbed motion (1.3) will be asymptotically stable under conditions (1.12).

2. The heavy gyroscope. Let us proceed to the question of the stability of the sphere under the action of moment (1.1), in which the center of gravity does not coincide with the point of suspension O. Let us assume that the center of gravity of the sphere is located on its dynamic axis of symmetry Oz at a distance z_0 from point O.

The equations of motion of the sphere

$$\begin{aligned} A\frac{dp}{dt} + (C - A)\,qr &= mgz_0\gamma' + K(\omega\gamma - p), \\ A\frac{dq}{dt} + (A - C)\,pr &= -mgz_0\gamma + K(\omega\gamma' - q), \end{aligned} \qquad C\frac{dr}{dt} = K(\omega\gamma'' - r) \tag{2.1}$$

when ω = const, admit of a stationary solution (1.3). By confining ourselves to the previous notations, we investigate the stability of state (1.3) with respect to p, q, r, γ, γ' and γ''. Let us introduce into consideration the functions

$$\Phi_1 = A\ (\xi^2 + \eta^2) + C\zeta^2 + 2C\omega\zeta + 2mgz_0\delta, \tag{2.2}$$
$$\Phi_2 = A\ (\xi\alpha + \eta\beta) + C\zeta\delta + C\ (\omega\delta + \zeta),\ \ \Phi_3 = \alpha^2 + \beta^2 + \delta^2 + 2\delta$$

Functions Φ_2 and Φ_3 coincide with the corresponding functions F_2 and F_3 of Section 1. It is easy to see that the derivative of the functions V_1

$$2V_1 = \Phi_1 - 2\omega\Phi_2 + (C\omega^2 - mgz_0)\ \Phi_3$$

taken relative to the equations of perturbed motion for (1.3), has the form (1.9). The function $2V_1$ will be a quadratic form

$$2V_1 = A\ (\xi^2 + \eta^2) + C\zeta^2 - 2\omega\ (A\xi\alpha + A\eta\beta + C\zeta\delta) + \\ +(C\omega^2 - mgz_0)\ \ (\alpha^2 + \beta^2 + \delta^2) \tag{2.3}$$

resulting from a more general form constructed by Chetaev [3] when $\lambda = -\omega$ and $\mu = 0$ (in the notations of [3]).

The conditions of sign-definiteness of function (2.3)

$$(A - C)\ \omega^2 + mgz_0 < 0, \qquad z_0 < 0 \tag{2.4}$$

by virtue of what was stated above, will be the conditions for the asymptotic stability of motion (1.3). As before, the coefficient K does not enter directly into the conditions for asymptotic stability if we determine the damping rate of the perturbed motions of the gyroscope axis. Inequalities (2.4) may hold for any relation between A and C. In case $C > A$ inequalities (2.4) are equivalent to

$$z_0 < 0 \tag{2.5}$$

but in case $C < A$ they reduce to the condition

$$(A - C)\ \omega^2 + mgz_0 < 0 \tag{2.6}$$

3. The case $\omega = \omega(t)$. We studied above the stability of the established motion of a sphere, rotating around the vertical with a velocity equal to the constant velocity ω of the bowl. As shown by system (1.2), in the case $\omega = \omega(t)$, the nonstationary motion of the sphere

$$p = q = 0, \qquad r = \omega\ (t), \qquad \gamma = \gamma' = 0, \qquad \gamma'' = 1 \tag{3.1}$$

does not occur. Motion (3.1), nevertheless, is possible in the presence of special corrective devices which create an auxiliary moment $M_z^{\,k}(t)$

around axis Oz. In every real gyroscopic system there are usually introduced corrective devices to ensure the realization of some designed motion; this justifies to a known degree the assumptions of the stated problem.

If $M_z^k(t)$ satisfy the condition

$$M_z^k(t) = C\omega^{\cdot}(t) \tag{3.2}$$

then the system of equations

$$A\frac{dp}{dt} + (C - A)\,qr = K(\omega\gamma - p), \qquad A\frac{dq}{dt} + (A - C)\,pr = K(\omega\gamma' - q), \qquad C\frac{dr}{dt} = K(\omega\gamma'' - r) + M_z^k(t) \tag{3.3}$$

admit of the particular solution (3.1). Retaining the adopted notations, we write the equations of perturbed motion corresponding to (3.1)

$$A\frac{d\xi}{dt} + (C - A)\,\eta\,(\omega + \zeta) = K(\omega\alpha - \xi), \qquad \frac{d\alpha}{dt} = \beta(\omega + \zeta) - \eta(1 + \delta)$$
$$A\frac{d\eta}{dt} + (A - C)\,\xi\,(\omega + \zeta) = K(\omega\beta - \eta), \qquad \frac{d\beta}{dt} = \xi(1 + \delta) - \alpha(\omega + \zeta)$$
$$C\frac{d\zeta}{dt} = K(\omega\delta - \zeta), \qquad \frac{d\delta}{dt} = \eta\alpha - \zeta\beta \tag{3.4}$$

As the Liapunov function let us consider the combination of functions

$$2V = F_1 - 2\omega(t)\,F_2 + C\omega^2(t)\,F_3 + \mu(t)\,F_4^2$$

where F_1, F_2 and F_3 are defined by (1.7) and $F_4 = \zeta$. The functions $\omega(t)$, $\mu(t)$ are assumed to be bounded, continuous, twice-differentiable on the infinite time interval $t \geqslant t_0$; moreover

$$\omega(t) > \omega^* > 0, \qquad \omega(t) \neq 0 \quad \text{when } t \geqslant t_0 \tag{3.5}$$

The function V which has been considered will be a quadratic form with variable coefficients

$$2V(t, \xi, \eta, \zeta, \alpha, \beta, \delta) = A(\xi^2 + \eta^2) + (C + \mu)\,\zeta^2 - $$
$$- 2\omega(A\xi\alpha + A\eta\beta + C\zeta\delta) + C\omega^2(\alpha^2 + \beta^2 + \delta^2) \tag{3.6}$$

The derivative of (3.6) taken relative to equations (3.4) will also be a quadratic form

$$V^{\cdot} = -K\,[\xi^2 + \eta^2 + \zeta^2 - 2\omega(\xi\alpha + \eta\beta + \zeta\delta) + \omega^2(\alpha^2 + \beta^2 + \delta^2)] -$$
$$- \omega^{\cdot}(A\xi\alpha + A\eta\beta + C\zeta\delta) + C\omega\omega^{\cdot}(\alpha^2 + \beta^2 + \delta^2) + \mu\frac{K}{C}\zeta(\omega\delta - \zeta) +$$
$$+ \frac{1}{2}\mu^{\cdot}\zeta^2 = -K\Psi(t, \xi, \eta, \zeta, \alpha, \beta, \delta) \tag{3.7}$$

On the basis of Liapunov's theorem on asymptotic stability, the unperturbed motion (3.1) turns out to be asymptotically stable if the function (3.6) is positive-definite and function (3.7) negative-definite in the sense of Liapunov (the requirement of the presence of an infinitely small upper bound in form V is satisfied under the given assumptions on the nature of functions $\omega(t)$ and $\mu(t)$). Thus, motion (3.1) is asymptotically stable if Sylvester's conditions are satisfied for the quadratic forms

$$\begin{aligned} 2V - \lambda_1(\xi^2+\eta^2+\zeta^2) - \lambda_2(\alpha^2+\beta^2+\delta^2) \\ \Psi - \lambda_3(\xi^2+\eta^2+\zeta^2) - \lambda_4(\alpha^2+\beta^2+\delta^2) \end{aligned} \qquad (3.8)$$

(λ_1, λ_2, λ_3 and λ_4 are positive numbers as small as desired). Each of these forms can be represented as a sum of three forms in the variables (ξ, α), (η, β) and (ζ, δ). The Sylvester conditions for these two-variable forms are

$$A > \lambda_1, \quad \begin{vmatrix} A-\lambda_1 & -\omega A \\ -\omega A & C\omega^2-\lambda_2 \end{vmatrix} > 0, \quad C+\mu > \lambda_1, \quad \begin{vmatrix} C+\mu-\lambda_1 & -\omega C \\ -\omega C & C\omega^2-\lambda_2 \end{vmatrix} > 0 \qquad (3.9)$$

$$1 > \lambda_3, \quad \begin{vmatrix} 1-\lambda_3 & -\omega+\dfrac{A}{2K}\dot\omega \\ -\omega+\dfrac{A}{2K}\dot\omega & \omega^2-\omega\dot\omega\dfrac{C}{K}-\lambda_4 \end{vmatrix} > 0, \quad \omega^2-\omega\dot\omega\frac{C}{K} > \lambda_3$$

$$\begin{vmatrix} \omega^2-\omega\dot\omega\dfrac{C}{K}-\lambda_3 & -\omega\left(1+\dfrac{\mu}{2C}\right)+\dfrac{C}{2K}\dot\omega \\ -\omega\left(1+\dfrac{\mu}{2C}\right)+\dfrac{C}{2K}\dot\omega & 1+\dfrac{\mu}{C}-\dfrac{\dot\mu}{2K}-\lambda_4 \end{vmatrix} > 0 \qquad (3.10)$$

In order to evaluate the basic results, let us analyze these inequalities when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. In this case inequalities (3.9) give

$$C > A, \qquad \mu > 0 \qquad (3.11)$$

Inequalities (3.10) reduce to

$$4K(A-C)\omega\dot\omega - A^2\dot\omega^2 > 0, \qquad \dot\mu\left(\omega^2-\omega\dot\omega\frac{C}{K}\right) + \frac{1}{2}K\left(\frac{\mu}{C}\omega+\frac{C}{K}\dot\omega\right)^2 < 0 \qquad (3.12)$$

The first relation in (3.12) gives

$$\frac{\dot\omega}{\omega} > \frac{4K(A-C)}{A^2} \qquad (3.13)$$

Let us now pass to the selection of function $\mu(t)$ constrained by inequalities (3.11) and (3.12). When

$$\mu = -\frac{C^2}{K}\frac{\dot\omega}{\omega}, \qquad \dot\mu < 0 \qquad (3.14)$$

the second relation in (3.12) is satisfied.

Taking (3.11) into account, we get the following bounds on ω:

$$\frac{\omega^{\cdot}}{\omega} < 0, \qquad \left(\frac{\omega^{\cdot}}{\omega}\right)^{\cdot} > 0 \tag{3.15}$$

By combining (3.13) and (3.15) we obtain inequalities to which conditions (3.9) and (3.10) reduce when $\lambda_1 = \ldots \lambda_4 = 0$

$$C > A,\ 0 > \frac{\omega^{\cdot}}{\omega} > -4K\frac{C-A}{A^2}, \qquad \left(\frac{\omega^{\cdot}}{\omega}\right)^{\cdot} > 0 \tag{3.16}$$

If the condition $\lambda_1 = \ldots = \lambda_4 = 0$ is eliminated from the hypothesis, then, as is not difficult to verify, inequalities (3.9) and (3.10) give the following sufficient conditions for the asymptotic stability of state (3.1):

$$C > A + \varepsilon_1,\ -\varepsilon_3 > \frac{\omega^{\cdot}}{\omega} > -4K\frac{C-A}{A^2} + \varepsilon_2 < 0, \qquad \left(\frac{\omega^{\cdot}}{\omega}\right)^{\cdot} > 0 \tag{3.17}$$

where ε_1, ε_2 and ε_3 are strictly positive numbers as small as desired. Inequalities (3.17) define a certain class of asymptotically stable motions (3.1). Conditions (3.17) were obtained from a consideration of a special kind of function (3.6) when the coefficients $\mu(t)$ were defined in accordance with (3.14). However, generally speaking, we can find other functions μ satisfying differential inequalities (3.12) which can widen the class of asymptotically stable motions (3.1). This aim can also be accomplished by considering along with (3.6) a Liapunov function of a more general form

$$2V^* = F_1 + 2\lambda F_2 - C\omega\lambda F_3 + \mu F_4^2 - 2C(\omega + \lambda) F_4$$

where $\lambda = \lambda(t)$ is some function satisfying conditions analogous to the corresponding conditions for $\mu(t)$.

Let us examine under what conditions an asymptotically stable motion (3.1) does occur in the case of heavy gyroscopes. This problem is solved with the aid of the quadratic form

$$2V_1 = \Phi_1 - 2\omega\Phi_2 + (C\omega^2 - mgz_0)\Phi_3 + \mu\Phi_4^2 \tag{3.18}$$

where Φ_1, Φ_2 and Φ_3 are defined in accordance with (2.2) and Φ_4 coincides with F_4. As is not difficult to see, the derivative of (3.18) relative to the equations of perturbed motion coincides with (3.7). Along with (3.6) in the given case we obtain

$$(C-A)\omega^2 > mgz_0, \qquad -\frac{K}{C}\frac{mgz_0}{C\omega^2 - mgz_0}\omega > \omega^{\cdot} > -4K\frac{C-A}{A^2}\omega, \quad \left(\frac{\omega^{\cdot}}{\omega}\right)^{\cdot} > 0$$

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